

Question Define normalizer of an element of a group.

Answer Prove that the normalizer $N(a)$ of $a \in G$ is a subgroup of G .

Defn: If $a \in G$, then $N(a)$, the normalizer of a in G is the set of all those elements of G which commute with a .

Symbolically, we write as

$$N(a) = \{x \in G : ax = xa\}.$$

Proof of theorem: Here, we have to prove that the normalizer $N(a)$ of $a \in G$ is a subgroup of G .

In order to prove this, let us proceed as follows:

We have

$$N(a) = \{x \in G : ax = xa\}.$$

Let $x_1, x_2 \in N(a)$.

Then $ax_1 = x_1a$, $ax_2 = x_2a$.

First of all, we show that $x_2^{-1} \in N(a)$.

We have

$$ax_2 = x_2a$$

$$\Rightarrow x_2^{-1}(ax_2)x_2^{-1} = x_2^{-1}(x_2a)x_2^{-1}$$

$$\Rightarrow x_2^{-1}a = ax_2^{-1}$$

$$\Rightarrow x_2^{-1} \in N(a)$$

Now we will show that $x_1x_2^{-1} \in N(a)$

We have,

$$a(x_1x_2^{-1}) = (ax_1)x_2^{-1} = (x_1a)x_2^{-1}$$

$$= x_1(ax_2^{-1}) = x_1(x_2^{-1}a) = (x_1x_2^{-1})a$$

$$\therefore x_1x_2^{-1} \in N(a).$$

Thus, $x_1, x_2 \in N(a) \Rightarrow x_1x_2^{-1} \in N(a)$.

$\therefore N(a)$ is a subgroup of G .

Hence the proof.

Note-1: It should be noted here that $N(a)$ is not necessarily normal subgroup of G .

Note-2: Since $ex = xe \forall x \in G$, therefore we have $N(e) = G$.

Note-3: If G is an abelian group and $a \in G$, then

$$xa = ax \forall x \in G.$$

Therefore $N(a) = G$.

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